On Non-Computable Functions

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The construction of non-computable functions used in this paper is based on the principle that a finite, non-empty set of non-negative integers has a largest element. Also, this principle is used only for sets which are exceptionally well-defined by current standards. No enumeration of computable functions is used, and in this sense the diagonal process is not employed. Thus, it appears that an apparently self-evident principle, of constant use in every area of mathematics, yields non-constructive entities.

I. INTRODUCTION

The purpose of this note is to present some very simple instances of non-computable functions. Beyond their simplicity, these examples throw light upon the following basic point. If a function \( f(x) \) is to serve as an example of a non-computable function, then \( f(x) \) must be well-defined in some generally accepted sense; hence the efforts to construct examples of non-computable functions reveal the general conviction that over and beyond the class of computable (general recursive) functions there is a much wider class, the class of well-defined functions. The scope of this latter class is vague; in some quarters, there exists a belief that this class will be defined some day in precise terms acceptable to all. The examples of non-computable functions to be discussed below will be well defined in an extremely primitive sense; we shall use only the principle that a non-empty finite set of non-negative integers has a largest element. Furthermore, we shall use this principle only for exceptionally well-defined sets; and thus our construction will rest upon considerations which occur constantly in every area of mathematics. It may be of interest to note that we shall not use an enumeration of computable functions to show that our examples are non-computable functions. Thus, in this sense, we do not use the diagonal process.

II. TERMINOLOGY

We shall use binary Turing machines (that is, Turing machines with the binary alphabet \( 0, 1 \)), in the sense of the excellent presentation of
The polynomial in the ring $\mathbb{Z}[x]$ is irreducible.

Consider a polynomial in $\mathbb{Z}[x]$ of degree $n > 1$.

To prove that a polynomial is irreducible, we need to show that it cannot be factored into non-constant polynomials in $\mathbb{Z}[x]$.

Let $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a polynomial in $\mathbb{Z}[x]$.

Assume $p(x)$ is reducible in $\mathbb{Z}[x]$, then $p(x)$ can be factored as $p(x) = q(x)\cdot r(x)$ where $q(x)$ and $r(x)$ are polynomials in $\mathbb{Z}[x]$ with degrees less than $n$.

Since $\deg(q(x)) < n$ and $\deg(r(x)) < n$, we have $\deg(q(x)) + \deg(r(x)) = \deg(p(x)) = n$.

However, this is a contradiction since the sum of the degrees of two polynomials is at most equal to the degree of their product.

Therefore, $p(x)$ is irreducible in $\mathbb{Z}[x]$.

As an example, consider the polynomial $f(x) = x^2 + 1$ in $\mathbb{Z}[x]$.

Since $\deg(f(x)) = 2$, it is irreducible in $\mathbb{Z}[x]$ because it cannot be factored into polynomials of lower degree.

Similarly, the polynomial $g(x) = x^3 + 2x + 1$ is irreducible in $\mathbb{Z}[x]$ since it cannot be factored into polynomials of lower degree.

In conclusion, irreducibility in $\mathbb{Z}[x]$ is a fundamental property of polynomials and plays a crucial role in algebraic number theory and other areas of mathematics.
\[
(x)A - < (x)f
\]

where \( (x)A < (x)f \).

(1) This is the number function (as specified in Section I). We shall

name this as the number function.
The page contains a mathematical text discussing computability and proof by contradiction. The text includes logical expressions and mathematical notation, typical of a discussion on non-computable functions and the halting problem. The content is complex and requires a background in theoretical computer science to fully understand.
7.1 Remark

Suppose that, for a certain integer \( n_0 \), we somehow succeeded in determining the exact value of \( N_e(n_0) \). From (13)–(15) it follows that we can then determine \( S(n_0) \) also, and hence finally \( \Xi(n_0) \). Various other comments will readily occur to the reader. For example, the easily proved inequality

\[ S(n) \leq (n + 1) \Sigma(5n) 2^{\Xi(5n)} \]

gives rise to some curious observations.

VIII. SUMMARY

Inspection of the preceding presentation shows that we used in our constructions only the following "principle of the largest element": If \( E \) is a non-empty, finite set of non-negative integers, then \( E \) has a largest element. This principle is used constantly, as a matter of course, in every field of mathematics. Our examples above show that this principle, even if applied only to exceptionally well-defined sets \( E \), may take us beyond the realm of constructive mathematics. Of course, common everyday experiences may be used to illustrate this sort of phenomenon. For example, when the writer wanted to find a certain highway on an automobile trip, he received the following directions from the foreman of a construction crew: "Drive straight ahead on this road; you will cross some steel bridges; and after you cross the last steel bridge, make a left turn at the next intersection." Luckily, the unsolvable problem implied by this advice was resolved by a member of the construction crew who volunteered the information that "after you cross the last steel bridge, there isn't another steel bridge until you reach Richmond, 130 miles away." The reader may find it amusing to verify, by detailed study of the excellent book of Kleene (Ref.), that this little story illustrates, in a concrete manner, some truly basic points in the theory of computable functions.

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REFERENCE